

Unique Continuation for a Class of Degenerate Elliptic Operators

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1. INTRODUCTION

During recent years, the study of the unique continuation property for solutions of second order elliptic equations has been receiving increasing attention from workers in both partial differential equations and geometry. For strictly elliptic and semilinear operators, great understanding has been brought [1–4]. But there are few results for degenerate and quasilinear ones. In this paper we prove a result on this problem by employing N. Garofalo and F. H. Lin's approach in [1, 2].

Let Ω be a connected open subset of \mathbb{R}^n , $n \geq 3$. A map $u: \Omega \rightarrow \mathbb{R}^s$, $s \geq 1$, $u \in W_{\text{loc}}^{2,2}(\Omega)$ satisfies the system of elliptic equations

$$Lu = \operatorname{div}(|u|^{p-2} \nabla u) - f(x, u) = 0, \quad p \geq 2. \quad (1)$$

Here f is a map: $\Omega \times \mathbb{R}^1 \rightarrow \mathbb{R}^s$. Our result is on the unique continuation property of u . Before stating it we need to recall the relevant definitions. A function $u \in W_{\text{loc}}^{2,2}(\Omega)$ is said to vanish of infinite order at $x_0 \in \Omega$ if for $r > 0$ sufficiently small,

$$\int_{\partial B_r(x_0)} |u|^{2(p-1)} = O(r^N), \quad \text{for all } N > 0. \quad (2)$$

Here and below we let $B_r(x)$ and $\partial B_r(x)$ denote the ball and sphere, respectively, with radius r and center x . The strong (weak, resp.) unique continuation property says if we know that u vanishes of infinite order at some point $x_0 \in \Omega$ (u vanishes on some open set $\Omega_0 \subset \subset \Omega$, resp.), then u vanishes identically on Ω . The operator L defined in (1) is said to have the strong (weak, resp.) unique continuation property if all $W_{\text{loc}}^{2,2}(\Omega)$ solutions of $Lu = 0$ have the corresponding property. We have the following result.

THEOREM 1. Assume

$$|f(x, u)| \leq K |u|^{p-1}, \quad K = \text{const.} \quad (3)$$

then the operator L defined in (1) has a strong unique continuation property, i.e., any $W_{\text{loc}}^{2,2}(\Omega)$ solution u of (1) which vanishes of infinite order at some point $x_0 \in \Omega$ must vanish identically on Ω .

2. PROOFS

Throughout we let $\bar{x} \in \Omega$, $d = \text{dist}(\bar{x}, \partial\Omega) > 0$, $B = B_r = B_r(\bar{x})$, $w = |u|^{p-2}$. Define

$$H(r) = \int_{\partial B_r(\bar{x})} |u|^{2p-2} = \int_{\partial B_r(\bar{x})} w^2 u^2 \quad (4)$$

$$I(r) = \int_{\partial B_r(\bar{x})} |u|^{2p-4} u u_\rho = \int_{\partial B_r(\bar{x})} w^2 u u_\rho. \quad (5)$$

Here $\rho = |x - \bar{x}|$, $u_\rho = \langle \nabla u, ((x - \bar{x})/\rho) \rangle$. We have

THEOREM 2. Assume that u is a $W_{\text{loc}}^{2,2}(\Omega)$ solution of (1). Then there exists a positive constant $C_1 = C_1(p, n, K)$ such that

$$|I(r)| \leq (C_1/r) H(r), \quad \text{for } 0 < r \leq r_0 \quad (6)$$

and

$$\iint_{B_r(\bar{x})} |u|^{2p-2} \leq C_2 (rH(r) + r^2 I(r)), \quad \text{for } 0 < r \leq r_0. \quad (7)$$

Here $C_2 = 4(n + 2(p-1)^2)/n^2$, $r_0 = \min[d/2, (1/C_2 K)^{1/2}, 1]$.

COROLLARY 1.

$$H(r)r^{-C_3} \geq H(\bar{r})\bar{r}^{-C_3}, \quad \text{for } 0 < r \leq \bar{r} \leq r_0 \quad (8)$$

here $C_3 = 2(p-1)C_1 + n - 1 > 0$.

Proof. Equality

$$H'(r) = [(n-1)/r] H(r) + 2(p-1) I(r), \quad 0 < r \leq r_0 \quad (9)$$

and (6) imply

$$(H(r)r^{-C_3})' \leq 0.$$

Hence we obtain (8) by integration.

Q.E.D.

COROLLARY 2. u cannot vanish of infinite order at \bar{x} unless $u \equiv 0$ in $B_{r_0}(\bar{x})$.

Proof. If u vanishes of infinite order at \bar{x} , then Corollary 1

$$H(r) \geq \frac{H(\bar{r})}{\bar{r}^{C_3}} r^{C_3}, \quad 0 < r \leq \bar{r} \leq r_0,$$

and (2) imply that $H(\bar{r}) = 0$ for all $\bar{r} \in (0, r_0]$. This fact and (7) show $u \equiv 0$ in $B_{r_0}(\bar{x})$. Q.E.D.

Proof of Theorem 1. Let x_0 be the point at which u vanishes of infinite order. We need to show that for any connected open subset Ω_0 , with

$$x_0 \in \Omega_0, \quad \Omega_0 \subset \subset \Omega, \quad \text{dist}(\Omega_0, \partial\Omega) > 0 \quad (10)$$

u vanishes identically on Ω_0 . Due to (10), Theorem 2 holds with the same $r_0 = r_0(\bar{x})$ for all $\bar{x} \in \Omega_0$.

Corollary 2 says $u \equiv 0$ in $B_{r_0}(x_0)$. But any $x' \in \Omega$ with $\text{dist}(x', x_0) = r_0/2$ is also a point at which u vanishes of infinite order, for $u \equiv 0$ in $B_{(1/4)r_0}(x')$. Hence $u \equiv 0$ in $B_{r_0}(x')$. With fix step $r_0/2$, we can reach any point \bar{x} in Ω_0 , with $u \equiv 0$ in $B_{r_0}(\bar{x})$. Q.E.D.

Proof of Theorem 2. Define

$$D(r) = \iint_{B_r(\bar{x})} |u|^{2p-4} |\nabla u|^2 = \iint_{B_r(\bar{x})} w^2 |\nabla u|^2.$$

An application of the divergence theorem to $|u|^{2p-2}(x - \bar{x})$ and Minkowski's inequality yield

$$\begin{aligned} \iint_{B_r(\bar{x})} |u|^{2p-2} &\leq C_4(rH(r) + r^2D(r)), \quad r < d, \\ C_4 &= 2(n + 2(p-1)^2)/n^2. \end{aligned} \quad (11)$$

Using the formula

$$\text{div}(w^2 u \nabla u) = wuf(x, u) + (p-1)w^2 |\nabla u|^2$$

we obtain

$$I(r) = (p-1)D(r) + \iint_{B_r} wuf(x, u). \quad (12)$$

Relations (12), (11), and (3) give

$$D(r) \leq [1/(p-1)][I(r) + C_4 KrH(r) + C_4 Kr^2 D(r)].$$

Take r_0 as in Theorem 2, then

$$C_4 K r^2 \leq 1/2 \leq p-1 - (p-1)/2 \leq p-1 - C_4 K r^2, \quad \text{for } r \leq r_0,$$

and

$$0 \leq D(r) \leq 2I(r) + rH(r) \leq 2I(r) + H(r), \quad r \leq r_0 \quad (13)$$

$$0 \leq rH(r) + D(r) \leq 2[I(r) + rH(r)] \leq 2(I(r) + H(r)), \quad \text{for } r \leq r_0 \quad (14)$$

Inequalities (11) and (13) yield (7).

From (7) we conclude that either

- (i) $H(r_0)$. Then $u \equiv 0$ in $B_{r_0}(\bar{x})$ by (7). In this case we get (6); or
- (ii) $H(r_0) \neq 0$. Hence there exists a constant $r_1 \in [0, r_0)$ such that

$$\begin{aligned} H(r) &= I(r) = 0, & 0 \leq r \leq r_1, & u \equiv 0 \text{ in } B_{r_1}(\bar{x}) \\ H(r) &> 0 & \text{for } r_1 < r \leq r_0. \end{aligned}$$

Now we need only to prove (6) for $r_1 < r \leq r_0$.

We will take r from $(r_1, r_0]$ in the proof of this theorem if there is no other statement. Note that

$$N(r) = \frac{rI(r)}{H(r)} \geq -r \geq -1$$

by (14). Therefore we need to show that function

$$z(r) = 1 + N(r) = \frac{H(r) + rI(r)}{H(r)} \geq 0$$

has a upper bound on $(r_1, r_0]$. We will prove this fact by using Lemma 1.

LEMMA 1. Assume that $l(r) \geq 0$ is a continuous function on $(a, b]$, and $\int_a^b l(r) dr \leq C$ for some constant C . If function $z(r) \geq 0$ satisfies

$$z'(r) \geq -l(r) z(r), \quad \text{for } a < r \leq b, \quad (15)$$

then $z(r) \leq e^c z(b)$; in particular, z is bounded from above.

Proof. Inequality (15) gives

$$\left(z(r) \exp \left\{ - \int_r^b l(t) dt \right\} \right)' \geq 0.$$

Hence, $e^{-c} z(r) \leq e^{-\int_r^b l(t) dt} z(r) \leq z(b)$.

We are going to check that $z(r) = 1 + N(r)$ satisfies (15) for some function l with the required property. That is, we need to prove

$$\begin{aligned} & l(r) H(r) + r(I'(r) H(r) - I(r) H'(r)) \\ & \geq -l(r)(rI(r) + H(r)) H(r), \quad r_1 < r \leq r_0. \end{aligned} \quad (16)$$

From (12), (9) we get

the left hand of (16)

$$\begin{aligned} & = (p-1) r \left(\int_{\partial B_r(\bar{x})} w^2 u^2 \int_{\partial B_r(\bar{x})} w^2 |\nabla u|^2 - 2I(r)^2 \right) \\ & \quad + H(r) \left(r \int_{\partial B_r(\bar{x})} w u f(x, u) - (n-2) I(r) \right). \end{aligned} \quad (17)$$

We cite a lemma which we will prove at the end of this paper.

LEMMA 2. *We have*

$$\begin{aligned} \int_{\partial B_r(\bar{x})} w^2 |\nabla u|^2 &= 2 \int_{\partial B_r} w^2 u_\rho^2 + [(n-2)/r] D(r) \\ &\quad - (2/r) \iint_{B_r} \rho w u_\rho \operatorname{div}(w \nabla u), \end{aligned}$$

for $0 < r < \operatorname{dist}(\bar{x}, \partial\Omega)$.

Using Lemma 2, (17), and (12) we obtain

the left hand of (16)

$$\begin{aligned} & = 2(p-1) r \left(\int_{\partial B_r} w^2 u^2 \int_{\partial B_r} w^2 u_\rho^2 - I(r)^2 \right) \\ & \quad - H(r) \left[\iint_{B_r} w u f(x, u) + 2(p-1) \right. \\ & \quad \times \left. \iint_{B_r} \rho w u_\rho f(x, u) - r \int_{\partial B_r} w u f(x, u) \right] \\ & = I_1 - H(r)(I_2 + I_3 - I_4) \end{aligned}$$

$$I_1 \geq 0 \quad (\text{Schwartz inequality}),$$

$$I_2(r) \leq C_4 K r (H(r) + r D(r)),$$

$$\begin{aligned}
I_3(r) &\leq 2(p-1) Kr \iint w^2 |uu_\rho| \\
&\leq (p-1) Kr \left((1/r) \iint |u|^{2p-2} + rD(r) \right) \\
&\leq 2(p-1) K(C_4+1) r(H(r) + rD(r)), \quad \text{by (7)} \\
-I_4 &\leq KrH(r).
\end{aligned}$$

Hence

$$\begin{aligned}
&\text{the left hand of (16)} \\
&\geq -C_5 rH(r)(H(r) + rD(r)) \\
&\geq -2C_5 rH(r)(rI(r) + H(r)).
\end{aligned}$$

Here in the last inequality we have used (13) and

$$\begin{aligned}
H(r) + rD(r) &\leq H(r) + 2rI(r) + rH(r) \\
&\leq 2(rI(r) + H(r))
\end{aligned}$$

and

$$C_5 = K(C_4 + 2(p-1)(C_4 + 1) + 1) > 0.$$

Therefore we get (16) with $l(r) = C_5 r$.

Proof of Lemma 2. Using the divergence theorem we obtain

$$\iint_{B_r} \rho w u_\rho \operatorname{div}(w \nabla u) = r \int_{\partial B_r} w^2 u_\rho^2 - \iint_{B_r} w \nabla u \cdot \nabla(\rho w u_\rho).$$

Now

$$\begin{aligned}
& - \iint_B w \nabla u \cdot \nabla(\rho w u_\rho) \\
&= - \iint_B w \nabla u [w u_\rho \cdot \nabla \rho + \rho u_\rho \nabla w + \rho w \nabla(u_\rho)] \\
&= - \iint_B [w^2 u_\rho^2 + \rho w u_\rho \nabla u \nabla w + (1/2) \rho w^2 (|\nabla u|^2)_\rho \\
&\quad + w^2 |\nabla u|^2 - w^2 u_\rho^2] \\
&= - \iint_{B_r} [(1/2) \rho (w^2 |\nabla u|^2)_\rho + w^2 |\nabla u|^2] \\
&= - (r/2) \int_{\partial B_r} w^2 |\nabla u|^2 + (n/2 - 1) \iint_{B_r} w^2 |\nabla u|^2.
\end{aligned}$$

Here in the second equality we have used

$$\begin{aligned} \iint \rho w^2 \nabla u \nabla(u_\rho) &= \iint \rho w^2 \sum_{i,j} u_{x_i} \left(u_{x_j} \cdot \frac{x_j - \bar{x}_j}{\rho} \right)_{x_i} \\ &= \iint \rho w^2 \left[\sum_{i,j} u_{x_i} u_{x_j} \frac{x_j - \bar{x}_j}{\rho} \right. \\ &\quad \left. + \sum_{i,j} u_{x_i} u_{x_j} \left(\frac{\delta_{ij}}{\rho} - \frac{(x_i - \bar{x}_i)(x_j - \bar{x}_j)}{\rho^3} \right) \right] \\ &= \iint \rho w^2 [(1/2)(|\nabla u|^2)_\rho + (1/\rho) |\nabla u|^2 - (1/\rho) u_\rho^2]; \end{aligned}$$

in the third equality

$$\begin{aligned} \rho w u_\rho \nabla u \cdot \nabla w &= (\rho/2)(w^2)_\rho |\nabla u|^2 \\ &= (\rho/2)(w^2 |\nabla u|^2)_\rho - (\rho/2) w^2 (|\nabla u|^2)_\rho; \end{aligned}$$

in the fourth equality we have used

$$\begin{aligned} \iint_{B_r(\bar{x})} \rho(w^2 |\nabla u|^2)_\rho &= \iint_{B_r(\bar{x})} \nabla(w^2 |\nabla u|^2) \cdot (x - \bar{x}) \\ &= r \int_{\partial B_r(\bar{x})} w^2 |\nabla u|^2 \\ &\quad - \iint_{B_r(\bar{x})} w^2 |\nabla u|^2 \operatorname{div}(x - \bar{x}). \end{aligned}$$

Therefore we complete the proof of Lemma 2.

Q.E.D.

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